A New Method for Calculating the Shear Stiffness of RC Beams with Web Diagonal Cracks

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Abstract
Investigation shows that long span concrete box girder bridges often suffer from cracking and deflection diseases simultaneously. A considerable number of the cracks are diagonal web cracks, which reflects the reduction of the shear stiffness of the structure. However, the current specifications only consider the adverse effects caused by flexural cracks and ignore the shear deflection. In addition, some scholars have found that the horizontal reinforcement of the web also limits the development of diagonal cracks. This paper carried out shear tests on I-shaped beams with different forms of reinforcement in the web. Then, a new calculation method is proposed to calculate the shear stiffness after diagonal cracking, which takes into account the influence of the vertical and horizontal reinforcement at the same time.

Keywords: shear stiffness; diagonal crack; truss model; horizontal reinforcement; stirrup.

1 Introduction
On one hand, experimental research on the shear load capacity of RC beams has been relatively mature while few experiments have been carried out on the shear deformation. On the other hand, current specifications only provide calculation methods that consider structural degradation stiffness due to flexural cracks. Mainstream calculation methods include the stiffness analytical
method (GB 50010-2010[1]) and the effective moment of inertia method (ACI 318-19[2], Eurocode 2004[3]). Additional deflection due to shear is considered in the minimum stiffness principle.

Debernardi P. G.[4], Zheng K. Q.[5], and Huang Z.[6] et al. conducted experiments on shear stiffness. The parameters include stirrup, shear span, and longitudinal steel reinforcement ratios. During the test, bending and shear deformation were separated using a displacement-meter grid. Xu D.[7] and Zhao Y.[8] focused on the shear performance of structures under different web reinforcements. Although the shear deformation was not directly measured, it can be found from the load-deflection curve that the configuration of the web horizontal steel bars can improve the shear capacity and shear stiffness to a certain extent.

This paper proposes a reconstructed truss model that integrates the variable angle truss model and the Mohr coordinated truss model for determining the characteristic stiffness in the shear limit state. The new model will consider the effects of stirrups and web horizontal reinforcements. On this basis, the change law of stiffness is obtained through experimental data. This will provide a new idea for solving the shear stiffness of RC beams.

2 Theoretical Model

2.1 Variable Angle Truss Model and Mohr Coordinated Truss Model

The truss model is a classic model used to solve the shear problem of RC beams. For the variable angle truss model (Figure 1), Leonhardt F.[9] gave the formula for solving the stiffness of the variable angle truss model

\[ K_V^V = \frac{1}{\text{csc}^2 \alpha \sec^2 \alpha + \tan^2 \alpha \frac{\pi n}{\rho_v} \frac{E_c b_w d_v}{n \rho_v}} \]  

\( \pi \) is the ratio of the elastic modulus of steel and concrete \( \left( \frac{E_s}{E_c} \right) \). \( \rho_v \) is the stirrup ratio. \( \alpha \) is the angle of the diagonal crack. \( b_w \) is the thickness of the web. \( d_v \) is the effective shear depth. Equation (1) assumes that the stiffness of the chord is infinite, which underestimates the shear deformation. If the influence of the chord is considered, the stiffness \( K_V^V \) should be calculated according to Equation (2). In this equation, \( \rho_l \) is the longitudinal steel reinforcement ratio.

\[ K_V^V = \frac{1}{\text{csc}^2 \alpha \sec^2 \alpha + \tan^2 \alpha \frac{\pi n}{\rho_v} \frac{E_c b_w d_v}{n \rho_l}} \]  


Figure 1. Variable Angle Truss Model

The Mohr coordinated truss model summarized by Hsu T. T. C.[10] satisfies both the force balance equation and the deformation coordination equation, while the constitutive relationship of the material satisfies Hooke's law. This model is applied to orthogonally reinforced concrete slab elements. The steel bars in both directions bear tension, the cracked concrete bears compression, and the
tension and compression on the section constitute the shear force. If these basic slab elements are pieced together into a beam, the truss model shown in Figure 2 is obtained. Similarly, if the shear stiffness of a segment is solved according to the force method, the stiffness of the Mohr coordinated truss model $K^C_y$ can be calculated:

$$K^C_y = \frac{1}{\csc^2 \alpha \sec^2 \alpha + \frac{\tan^2 \alpha}{\rho_w} \cot^2 \alpha} \frac{E_c b_w d_w}{\rho_h}$$  \hspace{1cm} (3)$$

$\rho_h$ is the web horizontal reinforcement ratio. In the denominator of Formula (3), the first term represents the deformation effect of concrete, the second term represents the deformation effect of the stirrup and the third term represents the deformation effect of the web horizontal reinforcement.

### 2.2 Determination of angle $\alpha$

Pan Z.[11] derived the crack angle of the truss in the B region, which is essentially to find the angle $\alpha$ of the Mohr coordinated truss model. As shown in Figure 3, a slab with orthogonal reinforcement satisfies the force balance equation:

$$\rho_x f^x_s = \tau \cot \alpha$$  \hspace{1cm} (4)$$

$$\rho_y f^y_s = \tau \tan \alpha$$  \hspace{1cm} (5)$$

$$f_2 = \tau (\cot \alpha + \tan \alpha)$$  \hspace{1cm} (6)$$

Figure 4 shows the deformation coordination conditions:

$$\tan^2 \alpha = \frac{\varepsilon_x + \varepsilon_2}{\varepsilon_y + \varepsilon_2}$$  \hspace{1cm} (7)$$

The two materials satisfy the constitutive equation of linear elasticity, then:

$$f^x_s = nE_c \varepsilon_x$$  \hspace{1cm} (8)$$

$$f^y_s = nE_c \varepsilon_y$$  \hspace{1cm} (9)$$

$$f_2 = E_c \varepsilon_2$$  \hspace{1cm} (10)$$

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**Figure 3. Balance of Forces and Mohr’s Circle of Stress**

**Figure 4. Deformation Coordination and Mohr’s Circle of Strain**
By combining Equations (4)-(10), the angle of the Mohr coordinated truss model $\alpha_C$ can be obtained:

$$ \tan \alpha_C = \left(1 + \frac{1}{n \rho_x} \right)^{\frac{1}{4}} $$

(11)

$\rho_x$ equals $\rho_h$ and $\rho_y$ equals $\rho_v$, then:

$$ \tan \alpha_C = \left(1 + \frac{1}{n \rho_h} \right)^{\frac{1}{4}} $$

(12)

Collins M. P. [12] first gave the form of Formula (12), but he replaced $\rho_h$ with $\rho_l$. This shows that in the variable angle truss model, the horizontal component of the shear force is borne by the longitudinal reinforcement. Hence, the angle of the variable angle truss model $\alpha_V$ is expressed as:

$$ \tan \alpha_V = \left(1 + \frac{1}{n \rho_l} \right)^{\frac{1}{4}} $$

(13)

### 2.3 Reconstructed Truss Model

The difference between the two types of trusses lies in which steel bars bear the horizontal tension generated by the shear force. However, both resist the horizontal tension. Therefore, a new model composed of the above two truss models is formed, as shown in Figure 5. The shear stiffness of the new model is weighted by the shear stiffnesses of the original models. The weight coefficients of the variable angle truss model and the Mohr coordinated truss model are recorded as $\mu_1$ and $\mu_2$ respectively. The shear stiffness of the reconstructed model $K_y^R$ is:

$$ K_y^R = \mu_1 K_y^V + \mu_2 K_y^C $$

(14)

$$ \mu_1 + \mu_2 = 1 $$

(15)

In this parallel spring group system, there is always $\gamma_V = \gamma_C = \gamma_R$, $\gamma_V$, $\gamma_C$ and $\gamma_R$ are the shear deformations of the variable angle truss model, Mohr coordinated truss model and the reconstructed model under one shear force. There are countless solutions for $\mu_1$ and $\mu_2$. If it is guaranteed that $K_y^R \rightarrow K_y^V$ when $K_y^C \rightarrow 0$ and $K_y^R \rightarrow K_y^C$ when $K_y^V \rightarrow 0$, then one of the answers is:

$$ \mu_1 = \frac{K_y^V}{K_y^V + K_y^C} $$

(16)

$$ \mu_2 = \frac{K_y^C}{K_y^V + K_y^C} $$

(17)

### 3 Comparative Experiment

The dimensions and reinforcement of the experimental beams are shown in Figure 6. The deformations in two shear spans are measured by the displacement-meter grids. More specific reinforcement information is shown in Table 1.
The concrete compressive strengths $f'_c$ of the three specimens are 58.7MPa, 47.2MPa and 55.3MPa. The curves of shear deformation developed with load obtained from the experiment are shown in Figure 7a). By comparing the results of T1 and T2, it can be found that increasing the stirrup ratio can increase the shear stiffness. The results of T1 and T3 are very close, partly due to the increase in shear stiffness caused by the increase in concrete strength.

Referring to the effective moment of inertia method for solving the bending stiffness, after the diagonal crack is generated, the degradation law of shear stiffness $K_v$ is expressed as:

$$K_v = K_e \left( \frac{V_{cr}}{V} \right) + K_y R [1 - \left( \frac{V_{cr}}{V} \right)] \quad (18)$$

$V_{cr}$ is the cracking shear force [2] and $K_e$ is the elastic shear stiffness. It can be seen from Equation (18) that $K_v \rightarrow K_e$ when $V \rightarrow V_{cr}$ and $K_v \rightarrow K_y$ when $V \rightarrow \infty$. Figure 7b)-d) shows the shear deformation prediction curves of the three beams based on this method. The curves are very close to the experimental results.

**Table 1. Reinforcement Information**

<table>
<thead>
<tr>
<th>Reinforcement Ratio</th>
<th>Stirrup [%]</th>
<th>Horizontal Reinforcement [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>T1</td>
<td>0.79</td>
<td>0.79</td>
</tr>
<tr>
<td>T2</td>
<td>2.18</td>
<td>0.79</td>
</tr>
<tr>
<td>T3</td>
<td>0.79</td>
<td>1.40</td>
</tr>
</tbody>
</table>

**Figure 7. Load-Shear Deformation Curves:**

- a) Experimental Data;
- b) Prediction for T1;
- c) Prediction for T2;
- d) Prediction for T3
4 Discussion

This paper proposes a new method for calculating the shear stiffness of RC beams during diagonal crack development. The method combines two truss models and refers to the effective moment of inertia method, which can fully consider the role of stirrups and web horizontal reinforcements in the shear resistance process.

However, the relevant experiments only used web reinforcements as variables, and the influence of other factors (longitudinal reinforcement ratio, shear span ratio) has not been considered. In follow-up research, further work will be carried out.

5 Conclusions

The conclusions drawn are as follows:

(1) Under certain conditions, the reduction in shear stiffness caused by diagonal cracks in the web should not be ignored.

(2) Increasing the reinforcement ratios of stirrups and web horizontal steel bars can improve the shear stiffness of RC beams.

(3) The reconstructed truss model, which combines the variable angle truss model and the Mohr coordinated truss model, can consider the effects of the two steel bars, providing a new idea for solving the problem of shear stiffness.

6 Acknowledgements

Thanks to Anhui Transport Consulting & Design Institute Co., Ltd. for funding this project. Thanks to Anhui Institute of Building Research & Design, Anhui Province Key Laboratory of Green Building and Assembly Construction for guidance and help with the experiment in this article. Engineers involved in assisting include Tengsheng Yue, Zhian Jiao, Jianpeng Wei, and Dan Xu.

7 References


